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A New Asymptotic Theory for the Periodically Forced Laser

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Sustained relaxation oscillations and irregular spiking have been observed in many periodically modulated lasers (2). These observations have been substantiated numerically by recent studies of the laser rate equations (3,4). In this paper, we propose a new asymptotic analysis of the laser equations which assumes that the laser oscillations correspond to relaxation oscillations. We identify a large parameter and construct these periodic solutions using perturbation techniques. We obtain the equations for the Poincare map and determine the first period doubling bifurcation.

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A NEW ASYMPTOTIC THEORY FOR THE PERIODICALLY FORCED LASER

FORMULATION

We consider the laser rate equations for a single mode, homogeneously broadened and periodically modulated laser. They consist of two ordinary differential equations for the intensity I and the inversion of population D. We investigate the effect of periodic modulations of the cavity decay rate of the form $\kappa(T') = \kappa_0(1 + \Delta\cos(\sigma T'))$ where Δ and σ are the amplitude and the frequency of the periodic modulations, respectively. In terms of dimensionless variables, the problem is then formulated as [1]:

$$\frac{dI}{dT} = 2I[-1 + AD - \Delta\cos(\omega T)] \tag{1}$$

$$\frac{dD}{dT} = \gamma[1 - D(1 + I)] \tag{2}$$

where T is defined as $T = \kappa_0 T'$. A is the pump parameter, γ is the loss rate for the population divided by the cavity decay rate κ_0 and $\omega = \sigma/\kappa_0$.

Our objective is to analyze the periodic solutions of the laser equations in the limit $\gamma \to 0$ $(A > 1)$. This limit applies for class B lasers which includes ruby, YAG, CO_2 and semiconductors lasers [2]. If $A > 1$ and $\Delta = 0$, the zero intensity solution is unstable and Eqs. (1) and (2) admit a non-zero intensity solution given by

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$$I_0 = A - 1 \text{ and } D_0 = 1/A. \tag{3}$$

A linear stability analysis of this steady state solution shows that it corresponds to a stable focus as $\gamma \to 0$. In this limit, small perturbations from the steady state are decaying on an $O(\gamma^{-1})$ time scale with $O(\gamma^{-1/2})$ period oscillations. This suggests to reformulate Eqs (1) and (2) in terms of the deviations from the steady state solution (3) and to introduce a new basic time $t = \gamma^{1/2}T$. These new equations are formulated in [1,3] and are given by

$$\frac{dx}{dt} = - y - \epsilon x - \epsilon_2 xy \tag{4}$$

$$\frac{dy}{dt} = (1 + y)[x - \delta\cos(\Omega t)] \tag{5}$$

where the new variables x, y, t, and the parameters ϵ, ϵ_2, δ, Ω are defined by

$$x = (D - D_0)A\left(\frac{2}{\gamma I_0}\right)^{1/2},\ y = (I - I_0)/I_0,\ t = (2\gamma I_0)^{1/2}T \tag{6}$$

and

$$\epsilon = A\left(\frac{\gamma}{2I_0}\right)^{1/2},\ \epsilon_2 = \left(\frac{\gamma I_0}{2}\right)^{1/2},\ \delta = \Delta\left(\frac{2}{\gamma I_0}\right)^{1/2},\ \Omega = \omega/(2\gamma I_0)^{1/2} \tag{7}$$

Note that ϵ and ϵ_2 are $O(\gamma^{1/2})$ small quantities. The parameters δ and Ω are two control parameters which we assume O(1).

Numerical studies of Eqs. (4) and (5) are given in [3,4]. These studies have shown that the bifurcation diagram of the periodically forced

laser admits several branches of periodic solutions appearing either from period doubling bifurcations or from saddle-node bifurcation points.

THE CONSERVATIVE SYSTEM

If $\delta = \epsilon = \epsilon_2 = 0$, Eqs. (4) and (5) reduces to a conservative system of equations given by

$$\frac{dX}{dt} = -Y \tag{8}$$

$$\frac{dY}{dt} = X(1 + Y) \tag{9}$$

This system is conservative and admits a one-parameter family of periodic solutions. A first integral is given by

$$C = Y - \ln(1 + Y) + \frac{1}{2}X^2 \tag{10}$$

where C is the constant of integration. For each $C \geq 0$, there exists a periodic orbit in the phase plane (X,Y). The orbit is surrounding the origin and is bounded below by the line $Y = -1$. An example is given in Figure 1a for $C = 4$. The broken line corresponds to $Y = -1$. Figure 1b shows $Y(t)$. Note that the oscillations of $X(t)$ are almost triangular and $Y(t)$ is nearly equal to $Y = -1$ except during a short interval of time. These properties of the periodic solutions becomes more dramatic as C becomes larger and will be used in our asymptotic analysis.

SMALL DAMPING AND SMALL AMPLITUDE FORCING

In [1] and [3], we proposed an asymptotic analysis of Eqs. (1) and (2) which is based on the simultaneous limit of small damping constants ($\epsilon \to 0$,

$\epsilon_2 = O(\epsilon)$) and small amplitude forcing ($\delta = O(\epsilon)$). In this limit, $2\pi n/\Omega$-periodic solutions ($n=1,2,3,..$) of Eq. (4) and (5) are constructed using a regular perturbation analysis in ϵ. We have found that there exist for each n two distinct branches of periodic solutions which appear as δ surpasses a critical value $\delta = \delta_n(\epsilon)$. The branching of these solutions from the limit point located at $\delta = \delta_n$ is called primary saddle node bifurcation in [3]. The branch of saddles emerging from this point is responsible for the irregular spiking observed in [2]. The asymptotic analysis based on the limit $\epsilon = O(\epsilon_2) \to 0$ is valid provided that the values $\delta = \delta_n(\epsilon)$ are $O(\epsilon)$ quantities. However, the exact numerical determination of the periodic solutions for specific small values of ϵ and ϵ_2 [3] indicate that the values $\delta = \delta_n$ is $O(1)$ as $n = 2, 3,..$ and increases as $n \to \infty$. Moreover, the $2\pi n/\Omega$-periodic solutions are large amplitude relaxation oscillations similar to the periodic solution of the conservative system (8) and (9) when C is large. We have noted that these relaxation oscillations already appear for the case $n = 1$ when $\delta = O(1)$. These observations suggest to analyze the periodic solutions of Eqs. (4) and (5) as relaxation oscillations. To this end, we propose a new asymptotic method based on the limit $C \to \infty$ keeping ϵ, ϵ_2 and δ fixed.

THE PERIODICALLY MODULATED CONSERVATIVE SYSTEM

In order to clearly differentiate the effects produced by the periodic modulations and the effects related to the damping coefficients ϵ and ϵ_2, we

consider the simplified problem with $\epsilon = \epsilon_2 = 0$. From Eqs. (4) and (5), we then find

$$\frac{dx}{dt} = -y \tag{11}$$

$$\frac{dy}{dt} = (1 + y)[x - \delta\cos(\Omega t)] \tag{12}$$

The bifurcation diagram of the $2\pi/\Omega$-periodic solutions (period 1 branch) and the $4\pi/\Omega$-periodic solutions (period 2 branch) has been analyzed numerically for $\Omega = 0.9$ using a continuation method. We have found that the bifurcation diagram for $\epsilon = \epsilon_2 = 0$ is a good approximation of the bifurcation diagram when $\epsilon \neq 0$ and $\epsilon_2 \neq 0$ [3] if δ is not too small. The period 1 branch starts at the origin of the amplitude vs δ bifurcation diagram and is S-shaped. Its left limit point is located at $\delta = 0$. The bifurcation diagram can be determined by a perturbation analysis treating Eqs. (11) and (12) as a weakly perturbed harmonic oscillator. The method is described in [1] and correctly predicts the S-shaped bifurcation diagram. The period 2 branch emerges from the period 1 branch at a period doubling bifurcation point ($\delta \approx 1.26$), is S-shaped and admits a right and a left limit point at $\delta \approx 3.61$ and $\delta = 0$, respectively.

From the linearized theory for the $2\pi n/\Omega$-periodic solutions, it can be shown that the periodic solutions are either neutrally stable (elliptic point) or unstable (saddle point). This suggests that Eqs. (11) and (12) is a conservative system of equations.

THE SINGULAR PERTURBATION ANALYSIS

We now propose to construct the periodic solutions using a singular perturbation method. As $C \to \infty$, the closed orbit in Figure 1a becomes more triangular and is characterized by a slow evolution near $y = -1$ and a quick large $O(C)$ pulse in y. This suggests to construct the periodic solution in two parts. The first part (called outer solution) corresponds to the slow evolution of the solution. The second part (called inner solution) describes the rapid change of both x and y.

Outer solution. The outer solution is characterized by $y \approx -1$. In first approximation, we obtain the following equations

$$\frac{dx}{dt} = 1 \text{ and } \frac{dy}{dt} = (1 + y)[x - \delta\cos(\Omega t)]. \tag{13}$$

We solve these equations with the initial conditions

$$x(t_0) = x_0 < 0, \ y(t_0) = y_0 > -1. \tag{14}$$

After integration, we obtain

$$x(t) = x_0 + (t - t_0) \tag{15}$$

and

$$y(t) = -1 + (y_0 + 1)e^{f(t)} \tag{16}$$

where

$$f(t) = x_0(t - t_0) + \frac{1}{2}(t - t_0)^2 - \frac{\delta}{\Omega}[\sin(\Omega t) - \sin(\Omega t_0)]. \tag{17}$$

Since $x_0 < 0$, $f(t)$ is negative near $t = t_0$ and $y + 1$ is exponentially small until $f(t)$ becomes positive. The critical time $t = t_1$ satisfies the condition $f(t) = 0$ or equivalently

$$x_0(t_1 - t_0) + \frac{1}{2}(t_1 - t_0)^2 - \frac{\delta}{\Omega}[\sin(\Omega t_1) - \sin(\Omega t_0)] = 0. \tag{18}$$

At $t = t_1$, $x(t)$ and $y(t)$ admit the following values

$$x(t_1) = x_1 = x_0 + (t_1 - t_0) \text{ and } y(t_1) = y_0. \tag{19}$$

However, as soon as $t > t_1$, $y(t)$ is increasing exponentially and the approximation $y \approx -1$ is no more valid. We now assume that x and y are changing rapidly and propose a new approximation of the solution valid for $t \approx t_1$.

Inner solution. The inner solution or inner layer solution describes the sudden increase and decrease of y and is characterized by the fact that t remains close to t_1. Thus, this solution satisfies in first approximation Eqs. (11) and (12) with $t = t_1$:

$$\frac{dx}{dt} = - y \tag{20}$$

$$\frac{dy}{dt} = (1 + y)\{x - \delta\cos(\Omega t_1)\}. \tag{21}$$

We solve these equations in the phase plane. A first integral is given by

$$C = y - \ln(1 + y) + \frac{1}{2}x^2 - \delta x\cos(\Omega t_1) \tag{22}$$

where C is the constant of integration. The assumption that the inner solution is characterized by large values of both x and y implies that C is large. We connect C to the value $x = x_1$ previously computed from the outer solution by using matching conditions. To this end, we rewrite (22) as

$$y + 1 = \exp[\frac{1}{2}x^2 - \delta x\cos(\Omega t_1) - C + y] \tag{23}$$

Matching implies that as $y \to -1$, $x \to x_1$. Since C is large, the condition for

a bounded solution requires that the expression in brackets is zero at $y = -1$ and $x = x_1$. Thus, $x = x_1$ is the positive root of the following quadratic equation

$$\frac{1}{2}x^2 - \delta x\cos(\Omega t_1) - (C + 1) = 0. \tag{24}$$

The variable y is rapidly increasing and then decreasing. From (21) and then from (22), we note that the maximum value of y appears at $x = \delta\cos(\Omega t_1)$ and is an O(C) quantity. On the other hand, the variable x is progressively decreasing from $x = x_1 > 0$ to $x = x_2 < 0$ where y approaches $y \approx -1$. The critical value $x = x_2$ now corresponds to the negative root of Eq. (24) and is related to x_1 by the following relation

$$x_2 = - x_1 + 2\delta\cos(\Omega t_1). \tag{25}$$

THE POINCARE MAP

In summary, we have found a map describing the periodic solution of the forced laser equations (11) and (12). This map connects the successive minimum or maximum values of x. We denote by $x = x_{2n} < 0$ and $x = x_{2n+1} > 0$, the minimum and maximum values of x corresponding to the times $t = t_n$ and t_{n+1}, respectively (n=0,1,2,..). From (18), (19) and (25), we have the following relations between x_{2n}, x_{2n+1}, t_n and t_{n+1}:

(1) $$x_{2n+1} = x_{2n} + (t_{n+1} - t_n) \tag{26}$$

(2) $$x_{2n}(t_{n+1} - t_n) + \frac{1}{2}(t_{n+1} - t_n)^2 - \frac{\delta}{\Omega}[\sin(\Omega t_{n+1}) - \sin(\Omega t_n)] = 0 \tag{27}$$

(3) $$x_{2n+2} = - x_{2n+1} + 2\delta\cos(\Omega t_{n+1}). \tag{28}$$

We illustrate our analysis by seeking particular solutions of Eqs. (26)-(29). We consider the cases of period 1 solutions and the period doubling bifurcation.

Period 1 solutions. Period 1 solutions satisfy the conditions

$$x_{2n+2} = x_{2n} \text{ and } t_{n+1} - t_n = \frac{2\pi}{\Omega}. \tag{29}$$

Inserting (29) into Eqs. (26)-(28) leads to the following relations between x_{2n}, x_{2n+1} and t_n:

$$x_{2n+1} = x_{2n} + \left(\frac{2\pi}{\Omega}\right), \tag{30}$$

$$x_{2n}\left(\frac{2\pi}{\Omega}\right) + \frac{1}{2}\left(\frac{2\pi}{\Omega}\right)^2 = 0, \tag{31}$$

$$x_{2n} = - x_{2n+1} + 2\delta\cos(\Omega t_n). \tag{32}$$

From (32) and then from (31), we find x_{2n} and x_{2n+1}:

$$x_{2n} = - \frac{\pi}{\Omega},\ x_{2n+1} = \frac{\pi}{\Omega} \tag{33}$$

Then from (32), we obtain

$$\cos(\Omega t_n) = 0 \tag{34}$$

which implies

$$t_n = - \frac{\pi}{2\Omega} + \frac{2n\pi}{\Omega}. \tag{35}$$

The period doubling bifurcation. Period 2 solutions satisfy the conditions

$$x_{2n+4} = x_{2n} \text{ and } t_{n+2} - t_n = \frac{4\pi}{\Omega}. \tag{36}$$

We introduce the first condition in Eqs. (26)-(28) and formulate the problem for x_{2n}, x_{2n+1}, x_{2n+3}, t_{n+1} and t_{n+2}. The analysis of these equations can be simplified if we introduce the variable τ defined as: $\tau = t_{n+1} - t_n - \frac{2\pi}{\Omega}$. If

$\tau = 0$, the period 2 solution becomes a period 1 solution. In terms of τ, the problem can be reduced to two equations for τ and t_n. Since the period doubling bifurcation occurs as $\tau \to 0$, we have analyzed these equations in this limit and have found the conditions

$$1 + \delta\Omega\sin(\Omega t_n) = 0 \text{ and } \cos(\Omega t_n) = 0. \tag{37}$$

These conditions admit the solution

$$t_n = -\frac{\pi}{2\Omega} + \frac{2n\pi}{\Omega} \text{ and } \delta = 1/\Omega. \tag{38}$$

For $\Omega = 0.9$, the period doubling bifurcation is located at $\delta = 1/\Omega = 1.1$. The exact numerical value is $\delta \approx 1.26$.

CONCLUSIONS

In this preliminary report, we have shown how to use a singular perturbation approach to obtain a discrete map for the driven conservative laser problem. Periodic solutions of period 1 and 2 are located and the position of the first period doubling bifurcation agrees with the position obtained numerically for the laser rate equations.

It is known that the manifolds of the saddle periodic orbit governs the global behavior of phase space such as chaos and chaotic bursting. Since our technique may determine periodic orbits which are close to these branches of saddle orbits, our laser map captures some of the global behavior observed in the full system. The singular perturbation technique used here will be applied to the damped laser problem to exploit other global features in a future paper.

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REFERENCES

1. T. Erneux, S.M. Baer and P. Mandel, Subharmonic bifurcation and bistability of periodic solutions in a periodically modulated laser, Phys. Rev. A35, 1165-1171 (1987).
2. N.B. Abraham, P. Mandel and L.M. Narducci, Dynamical Instabilities and Pulsations in lasers, Prog. in Opt. 25, 3-190 (1988).
3. I.B. Schwartz, Infinite primary saddle-node bifurcation in periodically forced systems, Phys. Lett. A 126, 411-418 (1988)
4. H.G. Solari, E. Eschenazi, R. Gilmore and J.R. Tredicce, Influence of coexisting attractors on the dynamics of a laser system, Opt. Comm. 64, 49-53 (1987).

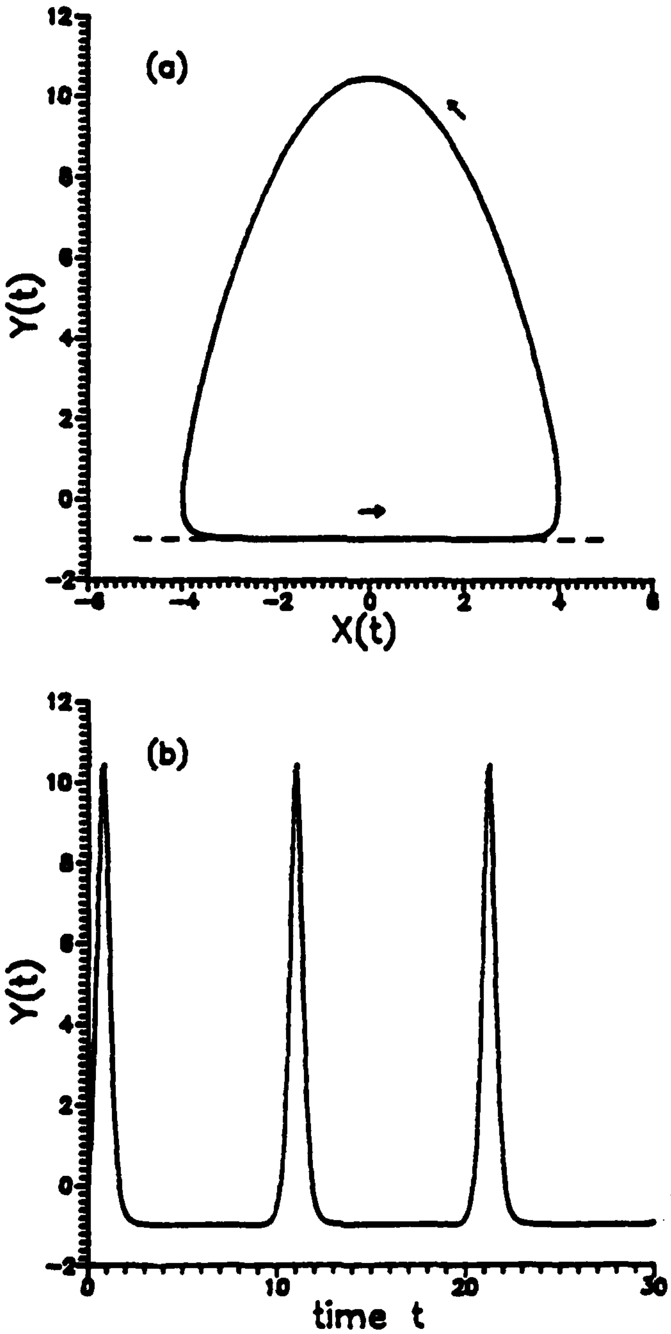

Fig. 1 — Periodic solution of the conservative system of equations (4) and (5)